ON CANONICAL CONFORMAL MAPS OF MULTIPLY CONNECTED REGIONS(1)

BY

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The first-named author has recently proved the following theorem [3]:

THEOREM 1. *Let D be a region of the extended z-plane whose boundary consists of mutually disjoint Jordan curves $B_1, B_2, \cdots, B_\mu$; $C_1, C_2, \cdots, C_\nu$, $\mu\nu \neq 0$. There exists a conformal map of D onto a region Δ of the extended Z-plane one to one and continuous in the closures of the two regions, where Δ is defined by*

$$(1) \qquad 1 < |T(Z)| < e^{1/\tau}, \qquad T(Z) \equiv \frac{A(Z-a_1)^{M_1}\cdots(Z-a_\mu)^{M_\mu}}{(Z-b_1)^{N_1}\cdots(Z-b_\nu)^{N_\nu}},$$

with

$$M_i,\ N_j,\ \tau > 0,\ \sum M_1 = \sum n_j = 1.$$

The locus $|T(Z)| = 1$ consists of μ mutually disjoint Jordan curves B_i^*, respective images of the B_i, which separate Δ from the a_i; the locus $|T(Z)| = e^{1/\tau}$ consists of ν mutually disjoint Jordan curves C_j^*, respective images of the C_j, which separate Δ from the b_j. The number τ is a conformal invariant of D and represents the period of the conjugate function of the harmonic measure of the set $\bigcup_{j=1}^{\nu} C_j$ with respect to D around the set of curves $\bigcup_{i=1}^{\mu} B_i$.

The region Δ represents a new canonical region for the conformal mapping of multiply connected domains; it has the property that the harmonic measure of the union of the curves C_j^* with respect to Δ is extendable harmonically to the entire Z-plane, except for one point in each of the components of the complement of Δ.

The object of the present paper is to demonstrate, using closely related methods, that the new theorem extends to the case of domains in which the sets $\bigcup_{i=1}^{\mu} B_i$ and $\bigcup_{j=1}^{\nu} C_j$ are made up of Jordan curves that are not necessarily disjoint. This result will later prove useful in the study of approximation by rational and by bounded analytic functions.

We begin with some necessary definitions:

A *Jordan configuration* is a finite collection of Jordan curves, no subset of which forms a closed cycle of Jordan curves; that is, every connected set of constituent curves is disconnected by the removal of any two points from any one of the curves.

Received by the editors March 7, 1958.

(1) This research was sponsored (in part) by the U. S. Air Force, Office of Scientific Research of the Air Research and Development Command.

A *multiple point* of a Jordan configuration is a point of the configuration which lies on more than one of the constituent curves. A multiple point is described uniquely by specifying the curves on which it lies, for if two different multiple points lie on each curve of a set of curves, the curves involved form a cycle, contrary to hypothesis.

THEOREM 2. *Let D be a region of the extended z-plane bounded by a Jordan configuration consisting of Jordan curves $B_1, B_2, \cdots, B_\mu, C_1, C_2, \cdots, C_\nu$, $\mu\nu\neq 0$, with $\bigcup_{i=1}^{\mu} B_i$ disjoint from $\bigcup_{j=1}^{\nu} C_j$. There exists a conformal map of D onto a region Δ of the extended Z-plane, one to one and continuous in the closures of the two regions, where Δ is defined by*

$$1 < |T(Z)| < e^{1/\tau},\quad T(Z) \equiv \frac{A(Z-a_1)^{M_1}\cdots(Z-a_\mu)^{M_\mu}}{(Z-b_1)^{N_1}\cdots(Z-b_\nu)^{N_\nu}},\ \text{with } M_i, N_j, \tau > 0, \tag{2}$$
$$\text{and } \sum M_i = \sum N_j = 1.$$

The locus $|T(Z)| = 1$ is a Jordan configuration composed of μ Jordan curves B_i^*, respective images of the B_i, which separate Δ from the a_i; the locus $|T(Z)| = e^{1/\tau}$ is a Jordan configuration composed of ν Jordan curves C_j^*, respective images of the C_j, which separate Δ from the b_j. The number τ is a conformal invariant of D and represents the period of the conjugate function of the harmonic measure of the set $\bigcup_{j=1}^{\nu} C_j$ with respect to D around the set of curves $\bigcup_{i=1}^{\mu} B_i$.

If a point a_i or b_j is at infinity the corresponding factor in (2) is to be omitted.

We may assume that D is bounded and that the curves B_i and C_j are analytic in the neighborhood of every point which is not a multiple point of the configuration, since that property may always be obtained by a preliminary conformal transformation of D.

Let D_k be a sequence of domains in the z-plane which satisfy:

(1) $\overline{D}_{k+1}\subset D_k$; $\overline{D}\subset D_k$ for all k;

(2) D_k is bounded by analytic curves B_i^k, C_j^k, separated from D by B_i, C_j respectively, $i=1, \cdots, \mu$, $j=1, \cdots, \nu$;

(3) $\lim_{k\to\infty} D_k = D$.

Let $u_k(z)$ be the function harmonic in D_k which takes the values 0, 1 on $\bigcup_{i=1}^{\mu} B_i^k$ and on $\bigcup_{j=1}^{\nu} C_j^k$ respectively, and let $u(z)$ be the function harmonic in D which takes the values 0, 1 on $\bigcup_{i=1}^{\mu} B_i$ and on $\bigcup_{j=1}^{\nu} C_j$ respectively.

The domains D_k are bounded by mutually disjoint Jordan curves and thus, by Theorem 1, for every D_k there exists a canonical conformal map $Z=f_k(z)$, taking D_k onto a domain Δ_k of the Z-plane, defined by $1<|T_k(Z)| < e^{1/\tau_k}$, where

$$T_k(Z) \equiv \frac{A_k(Z-a_{1k})^{M_{1k}}\cdots(Z-a_{\mu k})^{M_{\mu k}}}{(Z-b_{1k})^{N_{1k}}\cdots(Z-b_{\nu k})^{N_{\nu k}}},\ \text{with } M_{ik}, N_{jk}, \tau_k > 0, \tag{3}$$
$$\sum M_{i,k} = \sum N_{j,k} = 1,$$

and a_{ik}, b_{jk} are separated from Δ_k by B_i^{k*}, C_j^{k*}, the images under $f_k(z)$ of B_i^k, C_j^k respectively.

The boundary of D (thus of every D_k) consists of at least three Jordan curves, or else D falls into the category, already studied, of a domain with disjoint boundary curves. Following [3], select B_1, B_2, C_1 on different boundary components of D, and choose, by a suitable linear transformation of the Z-plane, $a_{1k}=0$, $a_{2k}=1$, $b_{1k}=\infty$ in $T_k(Z)$. As $k\to\infty$ there exists a partial sequence of k such that all the numbers A_k, a_{ik}, M_{ik}, b_{jk}, N_{jk}, τ_k approach limits (finite or infinite) A, a_i, M_i, b_j, N_j, τ; we will consider only this partial sequence hereforth. The functions $f_k(z)$ admit in D_k, thus also in D, the exceptional values 0, 1, ∞ and consequently form a normal family in D. We may therefore restrict ourselves further to a subsequence of values k for which the functions $f_k(z)$ approach a limit $f(z)$, uniformly on every compact in D.

Proposition 1. *$f(z)$ is not identically constant, hence defines a univalent map of D.*

Continuing with the method of [3] we choose an analytic Jordan curve γ in D which separates $\bigcup_{i=j}^{\mu} B_i$ from $\bigcup_{j=1}^{\nu} C_j$. The image of γ under $f_k(z)$ surrounds both $Z=0$ and $Z=1$, thus

$$\Delta_\gamma[\arg f_k(z)] = \Delta_\gamma[\arg (f_k(z) - 1)] = 2\pi,$$

which contradicts $f_k(z)\to c\neq\infty$ uniformly on γ. By use of an auxiliary linear transformation, $f_k(z)\to\infty$ is similarly seen to be impossible.

The above proof requires modification if D is doubly connected. In this case we choose a point α in D, and require $a_{1k}=0$, $b_{1k}=\infty$, $f_k(\alpha)=1$. The functions $f_k(z)$ admit in $D_k-\alpha$ and in $D-\alpha$ the exceptional values 0, 1, ∞. On a circumference with center α whose closed interior lies in D we have $\Delta \arg [f_k(z)-1]=2\pi$, so the proof can be completed as before. —For this modification the writers are indebted to Mr. Vincent Williams. A similar modification may be used in [3] to treat the case that D is doubly connected, and indeed was used [3, p. 142] in the proof of a limiting case of Theorem 1, as in the proof of Theorem 3 below.

Proposition 2. *$0<\tau<\infty$; τ represents the period around the set of curves, $\bigcup_{i=1}^{\mu} B_i$ of the conjugate function of the harmonic measure $u(z)$.*

The function $\tau_k \log |T_k[f_k(z)]|$ is defined and harmonic on D_k and takes the values 0, 1 on $\bigcup_{i=1}^{\mu} B_i^k$ and $\bigcup_{j=1}^{\nu} C_j^k$ respectively; it thus coincides everywhere in D_k with $u_k(z)$. Hence, denoting by n the interior normal,

$$\begin{aligned}\frac{1}{2\pi}\int_{\bigcup_{i=1}^{\mu} B_i^k} \frac{\partial u_k}{\partial n}\, ds &= \frac{\tau_k}{2\pi}\int_{\bigcup_{i=1}^{\mu} B_i^k} \frac{\partial}{\partial n}\log |T_k[f_k(z)]|\, ds \\ &= \frac{\tau_k}{2\pi}\int_{\bigcup_{i=1}^{\mu} B_i^{k*}} \frac{\partial}{\partial n}\log |T_k(Z)|\, ds = \tau_k. \end{aligned} \tag{4}$$

The function $u_k(z)-u_{k-1}(z)$ is harmonic everywhere in D_k and has values <0, >0 on $\bigcup_{i=1}^{\mu} B_i^k$ and $\bigcup_{j=1}^{\nu} C_j^k$ respectively. Let $v_k(z)$ be the function harmonic in D_k which takes the value 0 on $\bigcup_{i=1}^{\mu} B_i^k$ and which coincides with $u_k(z)-u_{k-1}(z)$ on $\bigcup_{j=1}^{\nu} C_j^k$. Then, letting $w_k(z)=v_k(z)-[u_k(z)-u_{k-1}(z)]$, we have $w_k(z)$ harmonic in D_k, $w_k(z)=0$ on $\bigcup_{j=1}^{\nu} C_j^k$, and $w_k(z)>0$ on $\bigcup_{i=1}^{\mu} B_i^k$. The function $w_k(z)$ may be extended harmonically beyond the curves $\bigcup_{j=1}^{\nu} C_j^k$; consequently $\partial w_k(z)/\partial n$ is defined on $\bigcup_{=1}^{\nu} C^k$ and, since $w_k(z)>0$ throughout D_k, is positive there. Thus

$$\int_{\bigcup_{j=1}^{\nu} C_j^k} \frac{\partial w_k}{\partial n} ds > 0. \tag{5}$$

The function $w_k(z)$ may be extended harmonically across the curves $\bigcup_{i=1}^{\mu} B_i^k$ since $v_k(z)$, $u_k(z)$, and $u_{k-1}(z)$ may all be. Hence $\partial w_k(z)/\partial n$ exists on $\bigcup_{i=1}^{\mu} B_i^k$, and since

$$\int_{\bigcup_{i=1}^{\mu} B_i^k + \bigcup_{j=1}^{\nu} C_j^k} \frac{\partial w_k}{\partial n} ds = 0$$

we have by (5)

$$\int_{\bigcup_{i=1}^{\mu} B_i^k} \frac{\partial w_k}{\partial n} ds < 0,$$

that is

$$\int_{\bigcup_{i=1}^{\mu} B_i^k} \frac{\partial}{\partial n}[u_k(z) - u_{k-1}(z)]ds > \int_{\bigcup_{i=1}^{\mu} B_i^k} \frac{\partial v_k(z)}{\partial n} ds. \tag{6}$$

But $v_k(z)=0$ on $\bigcup_{i=1}^{\mu} B_i^k$ and >0 in D_k, thus $\partial v_k/\partial n>0$ on $\bigcup_{i=1}^{\mu} B_i^k$, which implies, by (4), $\tau_k>\tau_{k-1}$, and in particular $\tau_k>\tau_1>0$.

Next let γ_0 be an analytic Jordan curve in D homotopic to $\bigcup_{i=1}^{\mu} B_i$. By virtue of (4) and of the harmonicity of $u_k(z)$ in D_k we have

$$\tau_k = \frac{1}{2\pi}\int_{\gamma_0} \frac{\partial u_k}{\partial n} ds.$$

By a theorem of Lebesgue [2], $\lim_{k\to\infty} u_k(z)=u(z)$, uniformly on every closed subset of D, thus also in some closed neighborhood of γ_0, whence

$$\tau = \lim_{k\to\infty} \tau_k = \frac{1}{2\pi}\int_{\gamma_0} \frac{\partial u}{\partial n} ds < \infty.$$

Thus $0<\tau<\infty$, and τ represents the period around the set of curves $\bigcup_{i=1}^{\mu} B_i$ of the conjugate of the harmonic measure $u(z)$.

PROPOSITION 3. M_r, $N_s>0$.

From the form (2) of $T_k(Z)$, and as in Proposition 2,

$$M_{rk} = \frac{1}{2\pi}\int_{B_r^{k*}} \frac{\partial \log |T_k(Z)|}{\partial n}\, ds = \frac{1}{2\pi\tau_k}\int_{B_r^k} \frac{\partial u_k(z)}{\partial n}\, ds.$$

Consider the function $v_r^k(z)$ harmonic in the region bounded by B_r^1, $\bigcup_{i\neq r} B_i^k$, and $\bigcup_{j=1}^{\nu} C_j^k$, which takes on the values 0 on B_r^1 and on $\bigcup_{i\neq r} B_i^k$, and 1 on $\bigcup_{j=1}^{\nu} C_j^k$. We have $v_r^k(z)>0$ on B_r^k, so that $v_r^k(z)-u_k(z)>0$ on B_r^k, $=0$ on $\bigcup_{i\neq r} B_i^k$ and on $\bigcup_{j=1}^{\nu} C_j^k$, whence

(7) $$\frac{\partial}{\partial n}[v_r^k(z) - u_k(z)] > 0, \quad z \text{ in } \bigcup_{i\neq r} B_i^k \text{ and in } \bigcup_{j=1}^{\nu} C_j^k, \text{ with } n \text{ the interior normal.}$$

Since $v_r^k - u_k$ is harmonic in D_k and in a neighborhood of the boundary of D_k,

$$\int_{\bigcup_{j=1}^{\mu} B_i^k + \bigcup_{j=1}^{\nu} C_j^k} \frac{\partial}{\partial n}[v_r^k - u_k]ds = 0,$$

whence by (7)

$$\int_{B_r^k} \frac{\partial}{\partial n}[v_r^k - u_k]ds < 0.$$

Thus

(8) $$\int_{B_r^k} \frac{\partial}{\partial n} u_k ds > \int_{B_r^k} \frac{\partial}{\partial n} v_r^k ds = \int_{B_r^1} \frac{\partial}{\partial n} v_r^k ds.$$

Next let $w_r(z)$ be the function harmonic in the region bounded by B_r^1, by $\bigcup_{i\neq r} B_i$, and by $\bigcup_{j=1}^{\nu} C_j^1$, which takes the values 0 on B_r^1 and on $\bigcup_{i\neq r} B_i$, and 1 on $\bigcup_{j=1}^{\nu} C_j^1$. Then the function $v_r^k(z)-w_r(z)$ is zero on B_r^1, >0 on $\bigcup_{i\neq r} B_i$ and on $\bigcup_{j=1}^{\nu} C_j^k$, so that $\partial[v_r^k - w_r]/\partial n > 0$ on B_r^1, whence

(9) $$\int_{B_r^1} \frac{\partial}{\partial n} v_r^k ds > \int_{B_r^1} \frac{\partial}{\partial n} w_r ds.$$

Combining (8) and (9) we obtain

$$M_{rk} = \frac{1}{2\pi\tau_k}\int_{B_r^k} \frac{\partial u_k}{\partial n} ds > \frac{1}{2\pi\tau_k}\int_{B_r^1} \frac{\partial w_r}{\partial n} ds.$$

The normal derivative $\partial w_r/\partial n > 0$ on B_r^1, and $1/\tau_k > 1/\tau > 0$, thus

(10) $$M_r > \frac{1}{\tau}\int_{B_r^1} \frac{\partial w_r(z)}{\partial n} ds > 0.$$

The identical argument applied to N_{sk} yields $N_s > 0$.

PROPOSITION 4. $a_i \neq b_j$, $i=1, \cdots, \mu$; $j=1, \cdots, \nu$.

We may find in D an analytic Jordan curve γ which separates $\bigcup_{i=1}^{\mu} B_i$ from $\bigcup_{j=1}^{\nu} C_j$, that is one for which

$$\frac{1}{2\pi i}\int_\gamma \frac{f_k'(z)dz}{f_k(z)-a_{ik}}=1,\qquad \frac{1}{2\pi i}\int_\gamma \frac{f_k'(z)dz}{f_k(z)-b_{jk}}=0.$$

Letting $k\to\infty$ in the above integrals we obtain

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)dz}{f(z)-a_i}=1\qquad \frac{1}{2\pi i}\int_\gamma \frac{f'(z)-dz}{f(z)-b_j}=0,$$

which implies that $a_i\neq b_j$.

If an analytic Jordan curve γ in D separates B_r and B_s, it follows as in the proof of Proposition 4 that the image of γ under the transformation $Z=f(z)$ separates the images of B_r and B_s. Otherwise expressed, if γ separates an annular neighborhood of B_r in D from such a neighborhood of B_s, then the image of γ separates the images in the Z-plane of the corresponding neighborhoods. We use here the following generalization (Carathéodory, [1, §120]) of Hurwitz's Theorem: if $f_k(z)$ is schlicht, $f_k(z)\to f(z)\not\equiv$const, uniformly in a region D_0, and if D_0^* is a closed subregion of D_0, then all points of $w=f(D_0^*)$ lie in $w=f_k(D_0)$ for k sufficiently large. Similarly for B_r and C_s.

Proposition 5. $a_r\neq a_s$, $b_r\neq b_s$ if $r\neq s$.

By a preliminary linear transformation of the Z-plane let all the a_i, b_j lie in a finite part of the Z-plane. If $a_r=a_s$, $r\neq s$, then given $\epsilon>0$ we may find k_0 so that $|a_{rk}-a_{sk}|<\epsilon$ for all $k>k_0$. Any two points a_{rk}, a_{sk} are separated from each other by a curve of the locus $|T_k(Z)|=1$; thus there will exist points Z_k with $|Z_k-a_{rk}|<\epsilon$ and $|T_k(Z_k)|=1$. However, because the points a_i are disjoint from the b_j (Proposition 4) the quantities $1/|Z-b_{jk}|^{N_{jk}}$ are all bounded independently of k if only k is sufficiently large and Z is in a sufficiently small neighborhood of a_{rk}. The $|Z-a_{ik}|^{M_{ik}}$ are also bounded uniformly in k under the same conditions, and since by the proof of Proposition 3 we have $M_{rk}>\alpha>0$, it follows that $|Z_k-a_{rk}|<\epsilon$ implies $|T_k(Z_k)|<G\epsilon^{M_{rk}}<G\epsilon^{\alpha}<1$ for ϵ sufficiently small, contradicting the requirement $|T_k(Z_k)|=1$. By the identical argument $b_r\neq b_s$ if $r\neq s$, and Proposition 5 is established. Let

$$T(Z)\equiv\frac{A(Z-a_1)^{M_1}\cdots(Z-a_\mu)^{M_\mu}}{(Z-b_1)^{N_1}\cdots(Z-b_\nu)^{N_\nu}};$$

we have $M_i, N_j>0$, $\sum M_i=\sum N_j=1$. On any closed set in the Z-plane which contains none of the points b_j, $j=1,\cdots,\nu$, the functions $T_k(Z)$ converge uniformly to $T(Z)$. If Δ is the domain in the Z-plane defined by $1<|T(Z)|<e^{1/\tau}$, the closure $\bar\Delta$ of Δ contains none of the points b_j, and thus in particular $T_k(Z)\to T(Z)$ uniformly on $\bar\Delta$. Thanks to Proposition 5, we can define Δ as the limit of Δ_k as $k\to\infty$. The domain Δ is connected, since no region of the

plane can be "pinched off" from Δ_k by the approach of boundary curves of Δ_k to the boundary curves of Δ (the latter are not disjoint). For if part of the plane were pinched off, the argument following Proposition 4, applied to an analytic Jordan curve γ in D which separates $\bigcup_{i=1}^{\mu} B_i$ from $\bigcup_{j=1}^{\nu} C_j$, establishes that such a region would be bounded wholly by points of the locus $|T(Z)|=1$ or wholly by points of the locus $|T(Z)|=e^{1/\tau}$, and would contain no pole or zero of $T(Z)$, which is impossible; the two loci mentioned can have no point in common.

We now consider the image under $Z=f(z)$ of a compact subdomain D^* of D. On every D^*, the function $f(z)$ is the uniform limit of $f_k(z)$, so that the image of D^* is the limit of the domains $f_k(D^*)$. But the domain $f_k(D^*)$ is included in the domain $f_k(D_k)=\Delta_k$, hence the limit $f(D^*)$ is included in the limit Δ of the domains Δ_k. We conclude that D is mapped by $f(z)$ onto a subdomain Δ_0 of Δ. As in [3] we next consider the inverse functions $f_k^{-1}(Z)$, defined and univalent in the regions Δ_k and thus, for k sufficiently large, in an arbitrary compact subdomain of Δ. The functions $f_k^{-1}(Z)$ are bounded in Δ_k, hence form a normal family in Δ, and we may restrict ourselves to a subsequence which approaches a limit $F(Z)$ on Δ, uniformly on every compact subset of Δ; $F(Z)$ is either univalent or identically constant on Δ. By the argument given above, Δ is mapped by $F(Z)$ onto a subset D_0 of D. For $Z_0 \in \Delta$, we have $\lim_{k\to\infty} f_k^{-1}(Z_0)=F(Z_0)\in D_0$, so that for k sufficiently large and Z_0 fixed all the points $f_k^{-1}(Z_0)$ lie in a compact subdomain of D. Then by Carathéodory's criterion for continuous convergence [1], we have $f[F(Z_0)]=\lim_{k\to\infty} f_k[f_k^{-1}(Z_0)]=Z_0$, hence $f(z)$ is the inverse function of $F(Z)$ for $Z\in\Delta$ and $z\in D_0$. This means that the image Δ_0 of D under $f(z)$ coincides with Δ, for every value Z_0 in Δ is assumed by $f(z)$ at the corresponding point $F(Z_0)\in D$.

PROPOSITION 6. *The locus $|T(Z)|=1$ is the limit of the loci $|T_k(Z)|=1$ as $k\to\infty$ and consists of μ Jordan curves B_i^*, definable as $B_i^*=\lim_{k\to\infty} B_i^{k^*}$; B_i^* separates Δ from the point a_i. Similarly, the locus $|T(Z)|=e^{1/\tau}$ is the limit of the loci $|T_k(Z)|=e^{1/\tau_k}$ and consists of ν Jordan curves C_j^*, definable as $C_j^*=\lim_{k\to\infty} C_j^{k^*}$; C_j^* separates Δ from the point b_j.*

Every critical point of $T(Z)$ (of which the total number is $\mu+\nu-2$) is a limit point of critical points of $T_k(Z)$, by Hurwitz's Theorem. Every point of the locus $|T(Z)|=1$ is a limit point of points of the loci $|T_k(Z)|=1$ as $k\to\infty$.

We show next that $B_i^*=\lim_{k\to\infty} B_i^{k^*}$ is a Jordan curve. B_i^* forms a part of the locus $|T(Z)|=1$. We prove first that a closed arc A of the locus $|T(Z)|=1$ not containing a critical point of $T(Z)$ cannot be the limit of more than one arc of the locus $|T_k(Z)|=1$. Let $N(A)$ be a neighborhood of A bounded by arcs of $|T(Z)|=1+\epsilon$, of $|T(Z)|=1-\epsilon$ (with $0<\epsilon<1/2$), and by arcs of two orthogonal trajectories of $|T(Z)|=$constant. If $N(A)$ is suitably chosen, throughout $N(A)$ we have $T'(Z)\neq 0$ and $|d\log T(Z)/dZ|\geqq 2\delta>0$, with

$\partial \log |T(Z)|/\partial s=0$ and $\partial \log |T(Z)|/\partial n \geqq 2\delta>0$, where the latter directions are taken respectively along and orthogonal to the loci $|T(Z)|=$constant. For k sufficiently large in $N(A)$ we have $\partial \log |T_k(Z)|/\partial n \geqq \delta$. The loci $|T_k(Z)|=\lambda$ approach uniformly the loci $|T(Z)|=\lambda$, in both position and direction throughout $N(A)$, so if $N(A)$ contains one arc of $|T_k(Z)|=1$ it cannot contain a second; thus A cannot be the limit of more than one arc of the locus $|T_k(Z)|=1$. This establishes that in tracing the curve B_i^* no arc of it is traversed more than once. Furthermore, every closed Jordan curve on which $|T(Z)|=1$ must separate Δ from one of the points a_i, otherwise in one of the regions into which the curve divides the plane the function $\log |T(Z)|$ is harmonic and has constant boundary values, which is impossible. Thus B_i^* cannot be made up of more than one closed curve. We conclude that $B_i^*=\lim_{k\to\infty} B_i^{k*}$ is a Jordan curve, separating Δ from the point a_i.

Proceeding identically for the curves C_j^* we obtain Proposition 6.

We have already shown, in establishing the connectedness of Δ, that no subset of the curves B_i^* and C_j^* can form a closed cycle. Thus the domain Δ is bounded by a Jordan configuration. The argument following Proposition 4 may be applied in D and in Δ to yield that a set of curves $B_{i_m}^*$ or $C_{j_n}^*$ forms a continuum if and only if the corresponding curves B_{i_m} or C_{j_n} form a continuum; we conclude that the boundary of Δ consists of a one to one transform of the curves (not points) of D, with each continuum transformed into a continuum. It will be noted that the locus $|T(Z)|=1$ has a critical point only where two or more of the curves B_i^* intersect, and the order of the critical point corresponds to the number of curves intersecting; otherwise an arc of that locus penetrates a region bounded by part of that locus and throughout which we have either $|T(Z)|<1$ or $|T(Z)|>1$. Similarly for the locus $|T(Z)|=e^{1/\tau}$.

It remains to study the boundary correspondence effected by the map $f(z)$.

PROPOSITION 7. *$Z=f(z)$ maps the curve B_r onto the curve B_r^*, and the curve C_s onto the curve C_s^*, $r=1, \cdots, \mu$; $s=1, \cdots, \nu$.*

The Jordan curve B_r has at most a finite number of points in common with the remaining boundary curves of D; denote these points by $p_1, \cdots, p_m$. Similarly, the curve B_r^* has at most a finite number of points in common with the remaining boundary curves of Δ; denote these by $P_1, \cdots, P_n$. The results of Carathéodory on boundary correspondence may be applied to the map $f(z)$ [1]: they establish that $Z=f(z)$ and $z=f^{-1}(Z)$ are one to one and continuous on the boundary, provided that a boundary point is construed as an equivalence class of arcs drawn in the region D or Δ from some fixed interior point to the boundary. In the present case this implies ordinary continuity and one to oneness of $f(z)$ in the neighborhood of any point z_0 for which both z_0 and $f(z_0)$ lie on only one boundary curve.

Let $v_k(z)$ be the function harmonic in D_k, equal to 0 on B_r^k and to 1 on the remaining boundary curves of D_k, and let $v(z)$ be the function harmonic and bounded in D, equal to 0 on B_r and to 1 on the remaining boundary curves of D, with the exception of the points $p_1, p_2, \cdots, p_m$, where it is undefined. Similarly, let $V_k(Z)$ be the function harmonic in Δ_k, equal to 0 on B_r^{k*} and to 1 on the remaining boundary curves of Δ_k, and let $V(Z)$ be the function harmonic and bounded in Δ, equal to 0 on B_r^* and to 1 on the remaining boundary curves of Δ, with the exception of the points $P_1, \cdots, P_n$, where it is undefined. We shall prove that $V_k(Z)\to V(Z)$, uniformly on any closed subset Δ_0 of Δ, indeed on any closed subset Δ_0 of $\bar{\Delta}$ containing no point P_j; an analogous proof establishes $v_k(z)\to v(z)$, uniformly on any closed subset D_0 of $\bar{D}$ containing no point p_j.

Let $V^\delta(Z)$ and $V_k^\delta(Z)$ be functions defined similarly to $V(Z)$ and $V_k(Z)$ respectively, with boundary values as before except that those values in the circle $|Z-P_j|\leqq\delta$ are now to be replaced by $|Z-P_j|/\delta$ on $\bigcup_{i\neq r} B_i^*$ and $\bigcup_{i\neq r} B_i^{k*}$.

Let R be twice the diameter of Δ, and set

$$Q^\delta(Z) = \sum_{j=1}^{n} \frac{\log R - \log|Z-P_j|}{\log R - \log\delta}, \qquad \delta < R,$$

whence $Q^\delta(Z)>0$ in Δ, and $\lim_{\delta\to 0} Q^\delta(Z)=0$, uniformly for Z in Δ_0.

Let Δ_0 be given, and let ϵ (>0) be arbitrary. Fix δ so small that we have $Q^\delta(Z)<\epsilon/3$ for Z in Δ_0, and that $\bigcup_{j=1}^{\nu} C_j^*$ and $\bigcup_{j=1}^{\nu} C_j^{k*}$ have no point in any $|Z-P_j|<\delta$. From a study of the respective boundary values we conclude, except for $Z=P_j$,

$$0 < V(Z) - V^\delta(Z) < Q^\delta(Z), \qquad Z \text{ in } \bar{\Delta},$$
$$0 < V_k(Z) - V_k^\delta(Z) < Q^\delta(Z), \qquad Z \text{ in } \bar{\Delta}_k,$$

whence for Z in Δ_0

$$\begin{aligned} &0 < V(Z) - V^\delta(Z) < \epsilon/3, \\ &0 < V_k(Z) - V_k^\delta(Z) < \epsilon/3. \end{aligned} \tag{11}$$

The boundary of Δ_k approaches that of Δ as $k\to\infty$, and the assigned boundary values of $V_k^\delta(Z)$ on Δ_k are continuous and approach those of $V^\delta(Z)$ on Δ uniformly, so by a theorem of Lebesgue [2] for k sufficiently large we have uniformly in $\bar{\Delta}$,

$$|V^\delta(Z) - V_k^\delta(Z)| < \epsilon/3;$$

this holds in particular for Z in Δ_0, whence by (11)

$$|V(Z) - V_k(Z)| < \epsilon, \qquad Z \text{ in } \Delta_0,$$

provided only k is sufficiently large. Proceeding analogously in the z-plane, we obtain

$$v(z) = \lim_{k\to\infty} v_k(z), \tag{12}$$

uniformly on any closed subset of $\overline{D}$ containing no point p_j.

The function $V_k[f_k(z)]$ is harmonic in D_k and coincides with $v_k(z)$ on the boundary curves of D_k, thus $V_k[f_k(z)] \equiv v_k(z)$. By (12) we have

$$\lim_{k\to\infty} V_k[f_k(z)] = v(z), \tag{13}$$

uniformly on any closed subdomain of D.

Given $z_0 \in B_r$, suppose $f(z_0) \notin B_r^*$. Then by the Carathéodory continuity of $f(z)$, there is a whole arc of B_r taken by $f(z)$ onto some B_q^*, $q \neq r$, and thus $V[f(z)] = 1$ on an arc of B_r. Consequently, $V[f(z)] \neq v(z)$ on a whole arc of the boundary of D, and thus there exists z_1 interior to D with $V[f(z_1)] \neq v(z_1)$. Since z_1 lies interior to D we have $Z_k = f_k(z_1) \to f(z_1) = Z_1$, and in a neighborhood of Z_1 the functions V_k converge uniformly to V; by Carathéodory's criterion for continuous convergence [1], it follows that $V_k(Z_k) \to V(Z_1)$. But this, together with (13), yields $v(z_1) = \lim_{k\to\infty} V_k(Z_k) = V(Z_1) = V[f(z_1)]$, and we have a contradiction.

The identical argument applied to the inverse functions $f^{-1}(Z)$ and $f_k^{-1}(Z)$ shows that the image of B_r under $f(z)$ covers all of B_r^*.

Proceeding as above for the curves C_s, we obtain Proposition 7.

PROPOSITION 8. *The multiple points $p_1, \cdots, p_m$ of B_r are mapped by $f(z)$ in a one to one way onto the multiple points $P_1, \cdots, P_n$ of B_r^*, $r = 1, \cdots, \mu$; similarly for the multiple points of C_s, $s = 1, \cdots, \nu$.*

Since $\bigcup_{i=1}^{\mu} B_i$ forms a Jordan configuration, a multiple point p_1 of B_r is described uniquely by specifying the constituent curves on which it lies. If $p_1 \in B_r \cap B_{i_1} \cdots \cap B_{i_q}$, by Proposition 7 every image $f(p_1) \in B_r^*$, $\in B_{i_1}^*, \cdots, \in B_{i_q}^*$, but, since $\bigcup_{i=1}^{\mu} B_i^*$ is likewise a Jordan configuration, $f(p_1)$ is thereby determined to be a single multiple point of B_r^*. Applying the identical argument to $f^{-1}(Z)$ we obtain the desired one to one correspondence.

Proceeding as above for the curves C_s, we obtain Proposition 8.

This completes the proof of the one to oneness and continuity of $f(z)$ on the boundary of D. For given $z_0 \in B_i$ not a multiple point, $f(z_0) \in B_i^*$ is also not a multiple point (Proposition 8), and continuity and one-to-oneness of $f(z)$ in a neighborhood of z_0 follow from Carathéodory continuity and one-to-oneness. If z_0 is a multiple point of B_i, $z_0 \in B_i, B_{i_2}, \cdots, B_{i_q}$, by Proposition 8 the point $f(z_0) \in B_i^*, B_{i_2}^*, \cdots, B_{i_q}^*$. Let γ be an arc in D joining a fixed interior point Q to a point ζ on the boundary, and let ζ trace the curves $B_i, B_{i_2}, \cdots, B_{i_q}$ in such a way that γ is continuously deformed and that every curve is covered only once; let the curves $B_i, B_{i_2}, \cdots, B_{i_q}$ be numbered sequentially in order of being transversed by ζ. Then for any two consecutive ones B_{i_m}, $B_{i_{m+1}}$ there exist arcs in D from Q to z_0 which have in their neighborhoods arcs joining Q both to points on B_{i_m} and to points on $B_{i_{m+1}}$. Carathéodory continuity now implies that $f(z)$ is continuous and one to one in any neighborhood of z_0. Theorem 2 is established.

As in [3], the validity of Theorem 2 may be extended to the limiting case in which the curves B_i are allowed to shrink to points:

THEOREM 3. *Let D be a region of the extended z-plane bounded by a Jordan configuration consisting of Jordan curves $C_1, \cdots, C_\nu$, let $\alpha_1, \cdots, \alpha_\mu$ be arbitrary distinct points of D, and let $M_1, \cdots, M_\mu$ be arbitrary positive numbers, with $\sum M_i = 1$. Then there exists a conformal map of D onto a region Δ of the extended Z-plane, one to one and continuous in the closures of the two regions where Δ is defined by*

$$|T(Z)| < 1, \quad T(Z) \equiv \frac{A(Z-a_1)^{M_1}\cdots(Z-a_\mu)^{M_\mu}}{(Z-b_1)^{N_1}\cdots(Z-b_\nu)^{N_\nu}}, \text{ with } N_j > 0 \sum N_j = 1.$$

The a_i are respective images of the α_i; the locus $|T(Z)| = 1$ is a Jordan configuration composed of ν Jordan curves C_j^*, respective images of the C_j, which separate Δ from the b_j.

The proof follows step by step that of Theorem 2, so that we may omit most of the details.

As before, take the curves C_j to be analytic in the neighborhood of any point not a multiple point of the configuration, and the domain D to be bounded. Consider a sequence of approximating domains D_k, defined as in Theorem 2, and bounded by disjoint contours C_j^k; by Theorem 3 of [3], the D_k may be mapped by $Z = f_k(z)$ onto a sequence of domains Δ_k in the Z-plane defined by

$$|T_k(Z)| < 1, \quad T_k(Z) \equiv \frac{A_k(Z-a_{1k})^{M_1}\cdots(Z-a_{\mu k})^{M_\mu}}{(Z-b_{1k})^{N_{1k}}\cdots(Z-b_{\nu k})^{N_{\nu k}}}, \quad \text{with } N_{jk} > 0,$$

$\sum N_{jk} = 1$, $a_{ik} = f_k(\alpha_i)$, and b_{jk} separated from Δ_k by $C_j^{k^*}$, the image under $f_k(z)$ of C_j^k.

We again choose $a_{1k} = 0$, $b_{ik} = 1$, $b_{2k} = \infty$ in $T_k(Z)$; the functions $f_k(z)$ admit in $D - \alpha_1$ the exceptional values 0, 1, ∞, thus form a normal family in $D - \alpha_1$. We restrict ourselves to a subsequence of values k for which $f_k(z)$ converges uniformly on every compact subdomain of $D - \alpha_1$ to $f(z)$, and for which all the numbers A_k, a_{ik}, b_{jk}, N_{jk} approach respective limits A, a_i, b_j, N_j.

Proposition 1 follows as before, establishing that $f(z)$ is not identically constant, hence a univalent map of $D - \alpha_1$. This implies univalence in D as well, for if $f(z_1) = f(z_2) = Z_0$, with $z_1, z_2 \in D$ and $z_1 \neq z_2$, the function $f(z)$ takes on every value in a neighborhood of Z_0 for z in a neighborhood both of z_1 and of z_2, contradicting the univalence of $f(z)$ in $D - \alpha_1$.

If D is simply connected, we modify this proof as before.

In Proposition 3, we have

$$N_{rk} = -\frac{1}{2\pi}\int_{C_r^{k*}} \frac{\partial}{\partial n}\log |T_k(Z)|\, ds = -\frac{1}{2\pi}\int_{C_r^k} \frac{\partial u_k(z)}{\partial n}\, ds,$$

where $u_k(z)$ is the function harmonic in D_k except for singularities $M_i \log |z-\alpha_i|$ in a neighborhood of each $z=\alpha_i$, and which takes on the value 0 for $z\in\bigcup_{j=1}^{\nu} C_j^k$. Precisely as before we obtain

$$N_{rk} > -\frac{1}{2\pi}\int_{C_r^1} \frac{\partial}{\partial n} w_r(z)ds > 0,$$

where $w_r(z)$ is the function harmonic in the domain bounded by C_r^1 and by $\bigcup_{j\neq r} C_j$, except for singularities M_i log $|z-\alpha_i|$ in a neighborhood of each $z=\alpha_i$, and which has the value 0 for $z\in C_r^1$, $\bigcup_{j\neq r} C_j$.

Proposition 4 follows with no change.

In Proposition 5 we have immediately $a_r\neq a_s$ for $r\neq s$, since $a_r=f(\alpha_r)$ and $a_s=f(\alpha_s)$ with $f(z)$ univalent, $z\in D$. The proof that $b_r\neq b_s$ if $r\neq s$ proceeds without change.

Letting

$$T(Z) \equiv \frac{A(Z-a_1)^{M_1}\cdots(Z-a_\mu)^{M_\mu}}{(Z-b_1)^{N_1}\cdots(Z-b_\nu)^{N_\nu}},$$

we have that the domains Δ_k: $|T_k(Z)|<1$ converge uniformly to Δ: $|T(Z)|<1$. By the same argument as in Theorem 2, Δ is connected and the locus $|T(Z)|=1$ is a Jordan configuration consisting of ν Jordan curves, possibly with multiple points, given by $C_j^*=\lim_{k\to\infty} C_j^{k*}$, with C_j^* separating Δ from b_j. The discussion which shows that D is mapped by $f(z)$ onto Δ follows precisely as before.

The discussion of Propositions 7 and 8 now applies without change.

We conclude that $f(z)$ maps the domain D conformally onto Δ, where Δ is defined by $|T(Z)|<1$, and is one to one and continuous in the closures of the two regions. Theorem 3 is established.

Under the conditions of Theorems 2 and 3, the maps whose existence is asserted are essentially unique. We outline the proof of

THEOREM 4. *Let D, defined by* (2) *with Z replaced by z, be a region of the extended z-plane bounded by a Jordan configuration consisting of Jordan curves $B_1, B_2, \cdots, B_\mu, C_1, C_2, \cdots, C_\nu$, $\mu\nu\neq 0$, with $\bigcup_{i=1}^{\mu} B_i$ disjoint from $\bigcup_{j=1}^{\nu} C_j$. Let B_i separate a_i from D, and C_j separate b_j from D. Let D be mapped conformally onto a region Δ of the extended Z-plane, one to one in the corresponding closed regions, where Δ is defined by*

$$1 < |T^*(Z)| < e^{1/\tau^*}, \qquad T^*(Z) \equiv \frac{A^*(Z-a_1^*)^{M_1^*}\cdots(Z-a_\mu^*)^{M_\mu^*}}{(Z-b_1^*)^{N_1^*}\cdots(Z-b_\nu^*)^{N_\nu^*}},$$

with M_i^*, N_j^*, $\tau^*>0$, $\sum M_i^* = \sum N_j^* = 1$, and let Δ be bounded by a Jordan configuration consisting of Jordan curves $B_1^*, B_2^*, \cdots, B_\mu^*, C_1^*, C_2^*, \cdots, C_\nu^*$, respective images of the B_i and C_j. Let B_i^* separate a_i^* from Δ and C_j^* separate b_j^* from Δ. Then the transformation $Z=Z(z)$ is defined by a linear transformation of the complex variable z.

The proof of Theorem 4 is quite similar to that of [3, Theorem 2]. The harmonic measure of $\bigcup_{j=1}^{\nu} C_j$ with respect to D is the transform of the harmonic measure of $\bigcup_{j=1}^{\nu} C_j^*$ with respect to Δ, and each is readily expressible in terms of $T(z)$ and $T^*(Z)$. Consideration of the variation of the conjugate of this harmonic measure on corresponding parts of the boundaries of D and Δ yields at once $\tau=\tau^*$, $M_i=M_i^*$, $N_j=N_j^*$. No region exterior to D bounded by a Jordan curve belonging, say, to the locus $|T(z)|=1$ can contain a critical point of $T(z)$, for any locus $|T(z)|=c$, $0<c<1$, divides the given region into precisely two subregions and cannot pass through a critical point of $T(z)$. It can now be established, as in [3], that the assumed map $Z=Z(z)$ of D onto Δ can be enlarged so as to map the extended z-plane one to one and conformally onto the extended Z-plane, hence is a linear transformation.

In Theorem 4, it is essential to assume that the given map defines a one to one correspondence of boundary points, not merely that $\bigcup_{i=1}^{\mu} B_i$ is transformed into $\bigcup_{i=1}^{\mu} B_i^*$. For let D be a doubly connected region with $\mu>1$, $\nu>1$. Then D admits a one to one map onto itself corresponding to any nontrivial rotation of the plane about 0 when D is mapped onto an annulus bounded by two circles with common center 0. Such a map of D onto itself carries $B=\bigcup_{i=1}^{\mu} B_i$ onto itself, and also $C=\bigcup_{j=1}^{\nu} C_j$, but need not be a linear transformation of the z-plane; it cannot be a linear transformation of the z-plane if we have $\mu=2$, $M_1\neq M_2$, for then the map does not transform the double point of B into a double point.

We add some general remarks as background material for Theorems 2 and 3 above, which in particular place in perspective the case that the given domain D is simply connected, and exhibit the canonical nature of the image domains Δ.

CASE I. Let D be a simply connected region bounded by a Jordan configuration composed of two Jordan curves B_1 and B_2 with precisely one multiple point. By the Riemann mapping theorem, D may be mapped conformally onto the interior of the unit circle, with the multiple point corresponding to two separate points on the circumference. Since there exists a map of the interior of the unit circle onto itself that carries three given points on the circumference onto any prescribed three points on the circumference which have the same order with respect to the positive sense of the boundary, it follows that any two domains D and D' of the type described may be mapped conformally one on the other by functions one to one and continuous in the closures of the two regions. The mapping function is not uniquely determined, but there is not enough freedom to ensure the correspondence of two arbitrary interior points. Letting $h(z)$ be the harmonic measure of B_1 with respect to D, and $H(Z)$ the transformed harmonic measure in D', an arbitrary point z_1 in D will necessarily be mapped onto a point of the locus $H(Z)=h(z_1)$ in D'; the mapping function may be fixed completely by requiring the point z_1 to correspond to a given point of that locus.

CASE II. Let D be a simply connected region bounded by a Jordan configuration composed of 3 Jordan curves B_1, B_2, B_3 with a single (common) multiple point. By the reasoning of Case I, any two such domains are mappable conformally one onto the other by a function one to one and continuous in the closures of the two domains. The mapping function is completely determined.

CASE III. Let D be a simply connected region of the z-plane bounded by any other Jordan configuration composed of Jordan curves $B_1, B_2, \cdots, B_m$ with multiple points $p_1, p_2, \cdots, p_r$; similarly, let D' be a simply connected region of the Z-plane bounded by a Jordan configuration composed of Jordan curves $C_1, C_2, \cdots, C_m$, with multiple points $q_1, q_2, \cdots, q_r$. Let P be any point interior to D, and consider the conjugate function $v_P(z)$ of the Green's function of D with pole at P. Let $\gamma_1, \gamma_2, \cdots, \gamma_s$ be all the level loci of $v_P(z)$ joining P to the multiple points p_j, $1 \leqq j \leqq r$, and let $0 \leqq \alpha_i < 2\pi$ be the angle with the horizontal, say, which γ_i forms at P. Similarly, let Q be any point interior to D', let $\delta_1, \cdots, \delta_s$ be the level loci of the conjugate of the Green's function of D' with pole at Q which join Q to the multiple points q_j, $1 \leqq j \leqq r$, and let β_i be the angle with the horizontal, say, which δ_i forms at Q. The map of the interior of D onto the interior of the unit circle which takes P onto zero is completely determined up to a rotation; when appropriately normalized, it associates with the points $p_1, \cdots, p_r$ the points $e^{i\alpha_1}, \cdots, e^{i\alpha_s}$. Similarly, the map, properly normalized, of the interior of D' onto the interior of the unit circle which takes Q onto zero associates with the points $q_1, \cdots, q_r$ the points $e^{i\beta_1}, \cdots, e^{i\beta_s}$. There exists a conformal map $Z=f(z)$ of D onto D', one to one and continuous in the closures of the two regions and taking $p_1, \cdots, p_r$ onto $q_1, \cdots, q_r$, if and only if there exists a map of the interior of the unit circle onto itself, taking the corresponding points on the unit circumference one on another, that is, renaming the points if necessary, one which takes $e^{i\alpha_1}, \cdots, e^{i\alpha_s}$ onto $e^{i\beta_1}, \cdots, e^{i\beta_s}$ respectively. A necessary and sufficient condition that there exist a conformal map of the unit circle onto itself which takes $e^{i\alpha_i}$ onto $e^{i\beta_i}$, $i=1, 2, 3, 4$, is the equality of the cross-ratios, one of which is easily seen to be equal to

$$X(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{\sin\left(\dfrac{\alpha_3-\alpha_1}{2}\right)\sin\left(\dfrac{\alpha_4-\alpha_2}{2}\right)}{\sin\left(\dfrac{\alpha_3-\alpha_2}{2}\right)\sin\left(\dfrac{\alpha_4-\alpha_1}{2}\right)}. \tag{14}$$

Thus a necessary and sufficient condition for the existence of the desired map of D onto D' is that

$$(15) \qquad X(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}) = X(\beta_i, \beta_{i+1}, \beta_{1+2}, \beta_{1+3}), \qquad 1 \leqq i \leqq s-3,$$

with X as defined in (14). We note that the condition is independent of the points P, Q which were chosen to be the poles of the respective Green's functions.

CASE IV. Let each of D and D' be a doubly connected region, of which one boundary component, B and B' respectively, is a single Jordan curve, and the other boundary component, C and C' respectively, is each a union of two Jordan curves with a single common point. Let $u(z)$ and $v(z)$ be conjugate functions of the harmonic measures of C with respect to D and of C' with respect to D' respectively. D and D' may be mapped one to one and conformally onto circular annuli R and R' respectively, centered at zero, with B and B' taken onto the inner boundary, which we may assume to be the unit circumference. Under this mapping, and multiple point of C corresponds to two distinct points p_1 and p_2 on the outer circumference of R; similarly, the multiple point of C' corresponds to two distinct points p_1' and p_2' on the outer circumference of R'. The functions $u(z)$ and $v(z)$ represent the argument of the image of a point $z \in D$ or D' under the map. There exists a conformal map of D onto D', one to one and continuous in the closures of the two regions, if and only if there exists a conformal map of R onto R' which takes the points p_1 and p_2 onto the points p_1' and p_2'. A necessary and sufficient condition for the above is that

(1) the moduli of D and D' are equal (the quantity $1/\tau$ of Theorem 2 gives the modulus of a doubly connected domain),

(2) the variation of $u(z)$ over an arc of C, both of whose endpoints are double point of C, equals the variation of $v(z)$ over an arc of C', both of whose endpoints are the double point of C'.

CASE V. Let each of D and D' be a doubly connected region of which one boundary component $B[B']$ is a single Jordan curve, and the other boundary component $C[C']$ is a Jordan configuration composed of three Jordan curves, all with a single common multiple point. Let $u(z)$ and $v(z)$ be as defined in Case IV. By the argument of Case IV, a necessary and sufficient condition for the existence of a conformal map of D onto D', one to one and continuous in the closures of the two regions, is

(1) the moduli of D and D' are equal,

(2) the variations of $u(z)$ over the three Jordan curves comprising C equal respectively the variations of $v(z)$ over the three Jordan curves comprising C', taken in the same sense with respect to D'.

CASE VI. Let each of D and D' be a doubly connected region of which each boundary component B and C $[B', C']$ is the union of two Jordan curves with a single common point. Let $u(z)$ and $v(z)$ be as defined in Case IV. A necessary and sufficient condition for the existence of a conformal map of D onto D', one to one and continuous in the closures of the two regions, with B and B' corresponding to each other, is

(1) the moduli of D and D' are equal,

(2) measured positively with respect to D from a fixed locus λ: $u(z)=$constant, not passing through a multiple point of B or C, the variations of $u(z)$ over the arcs of B and C from λ to the multiple points of B and C, and over the Jordan curves of B and C which do not meet λ, beginning and ending at the multiple points of B and C, equal respectively the variations of $v(z)$ measured positively with respect to D' from some corresponding fixed locus λ': $v(z)=$constant over the corresponding arcs of B' and C'.

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